NON-COMMUTING GRAPHS OF NILPOTENT GROUPS

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Abstract. Let G be a non-abelian group and $Z(G)$ be the center of G. The non-commuting graph Γ_G associated to G is the graph whose vertex set is $G \setminus Z(G)$ and two distinct elements x, y are adjacent if and only if $xy \neq yx$. We prove that if G and H are non-abelian nilpotent groups with irregular isomorphic non-commuting graphs, then $|G| = |H|$.

1. Introduction and results

Let G be a non-abelian group and $Z(G)$ be its center. The non-commuting graph Γ_G of G is a graph whose vertex set is $G \setminus Z(G)$ and two vertices x and y are adjacent if and only if $xy \neq yx$. The non-commuting graph of a group was first considered by Paul Erdős in 1975 [7]. Many people have studied the non-commuting graph (e.g., [1, 2, 3, 9, 10]). In [2] the following conjecture was put forward:

Conjecture 1.1 (Conjecture 1.1 of [2]). *Let G and H be two finite non-abelian groups such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.*

Conjecture 1.1 was refuted by an example due to Isaacs in [6], however it is valid whenever one of G or H is a non-abelian finite simple group [3] or whenever one of G or H has prime power order [1]. The counterexample given in [6] is a pair (G, H) of nilpotent non-abelian groups with regular non-commuting graph; recall that a graph is called regular if the degree of all vertices are the same, otherwise the graph is called irregular. It follows from a result of Ito [5] that a finite group with a regular non-commuting graph is a direct product of a non-abelian p-group for some prime p and an abelian group.

Here we study pairs (G, H) of non-abelian finite groups which provide a counterexample to Conjecture 1.1. It follows from the main result of [1], that if a pair (G, H) provides a counterexample then none of G and H are of prime power order. Here we prove that if a pair of non-abelian finite nilpotent groups provides a counterexample for Conjecture 1.1 then their non-commuting graphs must be regular.

Theorem 1.2. *Let G and H be two finite non-abelian nilpotent groups with irregular non-commuting graphs such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.*

We conjecture that the word "nilpotent" in Theorem 1.2 is sufficient for one of the groups G and H.

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2. Non-commuting graphs of nilpotent groups

A non-abelian group is called an AC-group if the centralizer of every non-central element is abelian. For a group G and an element $g \in G$, g^G denotes the conjugacy class of g in G.

Lemma 2.1. *Let G and H be two finite non-abelian groups. If $\phi : \Gamma_G \to \Gamma_H$ is a graph isomorphism and g is a non-central element of G, then the following hold:*
(1) $|G| - |Z(G)| = |H| - |Z(H)|$.
(2) $|G| - |C_G(g)| = |H| - |C_H(\phi(g))|$.
(3) $|C_G(g)| - |Z(C_G(g))| = |H| - |Z(C_H(\phi(g)))|$, *where $C_G(g)$ is not abelian.*
(4) If $C_G(g)$ is not abelian, then $\Gamma_{C_G(g)} \cong \Gamma_{C_H(\phi(g))}$.
(5) Suppose that $C_1 = C_G(g_1)$ and $C_i = C_{C_{i-1}}(g_i)$ for $i \geq 2$, where $g_1 \in G \setminus Z(G)$ and $g_i \in C_{i-1} \setminus Z(C_{i-1})$. Then there exists $k \in \mathbb{N}$ such that C_k is an AC-group.
(6) $|G| = |H|$ *if and only if* $|C_G(g)| = |C_H(\phi(g))|$ *if and only if* $|Z(G)| = |Z(H)|$.

Proof. It is straightforward. To prove (5), note that if the centralizer C_i is not an AC-group, then some proper centralizer in C_i is not abelian guaranteeing the existence of an element g_{i+1}. On the other hand, G is finite so the series $C_1 > C_2 > \cdots > C_i > \cdots$ will eventually terminate in an AC-group. □

Lemma 2.2 (see e.g. Theorem 2.1 of [1])**.** *Let G be a finite non-abelian group and H be a group such that $\Gamma_G \cong \Gamma_H$. Then the following hold:*
(1) $|C_H(h)|$ *divides* $(|g^G| - 1)(|Z(G)| - |Z(H)|)$ *for any $g \in G \setminus Z(G)$ and $h = \phi(g)$, where ϕ is any graph isomorphism from Γ_G to Γ_H.*
(2) If $|Z(G)| \geq |Z(H)|$ and G contains a non-central element g such that $|C_G(g)|^2 \geq |G| \cdot |Z(G)|$, then $|G| = |H|$.

We need the following result concerning a number theoretic conjecture due to Goormaghtigh.

Theorem 2.3 (see e.g. Theorem 1.3 of [4])**.** *Let x, y, m, n be integers such that $y > x > 1$ and $m, n > 1$. Then the following equation has at most one pair (m, n) of solution for every fixed pair (x, y):*

$$\frac{y^n - 1}{y - 1} = \frac{x^m - 1}{x - 1}.$$

Theorem 2.4. *Let G be a nilpotent group with at least two distinct non-abelian Sylow subgroups. Suppose also that H is any non-abelian group such that $|Z(G)| \geq |Z(H)|$ and $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.*

Proof. Suppose $G = P \times Q \times S$, where P and Q are non-abelian Sylow p, q-subgroups of G such that $p \neq q$ and S is a subgroup of G. If $x \in P \setminus Z(P)$ and $y \in Q \setminus Z(Q)$, then $|Z(G)| < |C_P(x)||C_Q(y)||S|$. Therefore

$$|G||Z(G)| < |C_P(x)||C_Q(y)||P||Q||S|^2 = |C_G(x)||C_G(y)|.$$

It follows that

$$|G||Z(G)| < \max\{|C_G(x)|^2, |C_G(y|^2\}.$$

Now, Lemma 2.2(2) completes the proof. □

Corollary 2.5. *Let G and H be two nilpotent groups each of which have at least two non-abelian Sylow subgroups. If $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.*

Both groups G and H in the counterexample of Conjecture 1.1 due to Isaacs in [6] have the same shape, that is, they are direct products of a non-abelian group of prime power order P and a non-trivial abelian group A such that $\gcd(|P|, |A|) = 1$ and all non-trivial conjugacy class sizes of G or H have equal order. The latter property was first studied by Ito [5] and we want to prove Theorem 1.2 for all nilpotent groups except those satisfying the latter shape.

3. Proof of Theorem 1.2

Now, we prove Theorem 1.2 in four cases. In this section G and H are finite non-abelian nilpotent groups with irregular non-commuting graphs and $\phi : \Gamma_G \to \Gamma_H$ is a graph isomorphism. By Corollary 2.5, we may assume that G has exactly one non-abelian Sylow subgroup. If G is of prime power order, the main result of [1] implies that $|G| = |H|$. Thus we may assume $G = P \times A$, where P is a non-abelian Sylow p-subgroup of G and A is a non-trivial abelian subgroup whose order is prime to p. Also, set $|P| = p^n$ and $|Z(P)| = p^r$.

Case (a): $H = P_1 \times B$ for some non-abelian Sylow p-subgroup P_1 of H and for some non-trivial abelian subgroup B of H.
We use the following notation: $|P_1| = p^m$, $|Z(P_1)| = p^s$ and $\phi(g_i) = h_i$, where $g_1, \ldots, g_k$ are non-central elements of G chosen from conjugacy classes of G with pairwise distinct sizes such that $|{g_i}^G| = p^{a_i}$ and $|{h_i}^H| = p^{b_i}$ and $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_k$. Notice that $k \geq 2$, since Γ_G and Γ_H are irregular.
Since $\Gamma_G \cong \Gamma_H$,

$$|A|p^r(p^{n-r} - 1) = |B|p^s(p^{m-s} - 1) \tag{1}$$

$$|A|p^{n-a_i}(p^{a_i} - 1) = |B|p^{m-b_i}(p^{b_i} - 1) \tag{2}$$

for every $1 \leq i \leq k$. Equation (1) implies that $r = s$ and equation (2) implies that $n - a_i = m - b_i$. Since Γ_G is not regular, graph isomorphism implies that

$$|A|(p^{n-a_1} - p^{n-a_2}) = |B|(p^{m-b_1} - p^{m-b_2}). \tag{3}$$

Therefore $|A| = |B|$. Now, equation (2) implies that $a_1 = b_1$. Hence $|P| = |P_1|$.

Case (b): $H = P_1 \times X$, where P_1 is a non-abelian Sylow p-subgroup of H and X is an arbitrary group such that $\gcd(p, |X|) = 1$.
Suppose H is a minimal counterexample. Also suppose by way of contradiction that X is a non-abelian group. Then P_1 and X are AC-group. Let $x \in X \setminus Z(X)$. Then $C_H(x) = P_1 \times B$, where $B \subseteq X$ is an abelian subgroup of X. Therefore Case (a) implies that $|C_H(x)| = |C_G(\phi^{-1}(x))|$. Since $\Gamma_G \cong \Gamma_H$, we have $|G| = |H|$. Now, $|G| = |H|$ implies that $|P| = |P_1|$. Set $|C_G(\phi^{-1}(x))| = p^{n-\alpha}|A|$ for some integer $1 < \alpha < n$. By graph isomorphism, we have

$$(p^n - p^{n-\alpha})|A| = p^n(|X| - |C_X(x)|).$$

The largest p-power dividing the right-hand side of the equation is $\geq p^n$ and the left is $p^{n-\alpha}$. This is a contradiction. Hence X is abelian and Case (a) completes the proof.

Case (c): $H = Q_1 \times X$**, where** Q_1 **is a Sylow** q_1**-subgroup of** H **and** X **is a non-abelian nilpotent group.**

If $p = q_1$, then Case (b) completes the proof. We claim that $p \neq q_1$ is not possible. Let H be a minimal counterexample. Therefore Q_1 and X are AC-groups. By the characterization of AC-groups [8], a nilpotent AC-group is a direct product of a non-abelian group of prime power order and an abelian group. Therefore $X = Q_2 \times B$, where Q_2 is a non-abelian q_2-group for some prime q_2, B is an abelian group and $\gcd(|Q_2|, |B|) = 1$. Let $h_i \in Q_i \setminus Z(Q_i)$ for $i \in \{1, 2\}$. Also, set $\phi^{-1}(h_i) = g_i$ for $i \in \{1, 2\}$ and $|C_G(g_i)| = |A|p^{n-a_i}$, where $1 < a_i < n$. If $q_2 = p$, then again Case (b) implies that $Q_1 \times B$ is an abelian group. This is a contradiction. Therefore $p \neq q_1, q_2$.

We have $C_H(h_1) = C_{Q_1}(h_1) \times Q_2 \times B$ and $C_H(h_2) = Q_1 \times C_{Q_2}(h_2) \times B$ and $Z(C_H(h_1)) = C_{Q_1}(h) \times Z(Q_2) \times B$ and $Z(C_H(h_2)) = Z(Q_1) \times C_{Q_2}(h_1) \times B$. So $Z(H) \subsetneqq Z(C_H(h_i))$. Therefore graph isomorphism implies that $Z(G) \subsetneqq Z(C_G(g_i))$. Set $|Z(C_G(g_i))| = |A|p^{d_i}$ for $i \in \{1, 2\}$ and $|Z(G)| = |A|p^r$. It is clear that $d_i > r$. Now, $\Gamma_G \cong \Gamma_H$ and $\Gamma_{C_G(g_i)} \cong \Gamma_{C_H(h_i)}$ for $i \in \{1, 2\}$ imply that

$$|C_H(h_2)| - |Z(C_H(h_2))| = (|Q_1| - |Z(Q_1)|)|C_{Q_2}(h_2)||B| = |A|(p^{n-a_2} - p^{d_2}) \tag{4}$$

$$|Z(C_H(h_1))| - |Z(H)| = (|C_{Q_1}(h_1)| - |Z(Q_1)|)|Z(Q_2)||B| = |A|(p^{d_1} - p^r) \tag{5}$$

$$|H| - |C_H(h_1)| = (|Q_1| - |C_{Q_1}(h_1)|)|Q_2||B| = |A|(p^n - p^{n-a_1}). \tag{6}$$

Since $|B|(|Q_1| - |Z(Q_1)|) = |B|(|Q_1| - |C_{Q_1}(h_1)|) + |B|(|C_{Q_1}(h_1)| - |Z(Q_1)|)$, by equation (5) and (6) the largest p-power dividing the right-hand side of the latter equation is p^r and by equation (4) the largest p-power dividing the left hand side is p^{d_2}. This is a contradiction.

Case (d): $H = Q \times B$**, where** Q **is a non-abelian Sylow** q**-subgroup for some prime** $q \neq p$ **and** B **is a non-trivial abelian subgroup.**

Suppose by way of contradiction that $|G| \neq |H|$. Since Γ_G is not regular, there exist $g_1, g_2 \in G \setminus Z(G)$ such that $|g_1^G| = p^{a_1} \neq p^{a_2} = |g_2^G|$. Set $|Q| = q^m$, $|Z(Q)| = q^s$, $\phi(g_i) = h_i$ for $i \in \{1, 2\}$ and $|h_i^H| = q^{b_i}$. Since $\Gamma_G \cong \Gamma_H$,

$$|A|(p^n - p^r) = |B|(q^m - q^s) \tag{7}$$

$$|A|(p^{n-a_i} - p^r) = |B|(q^{m-b_i} - q^s). \tag{8}$$

If $u = \gcd(a_1, a_2, n - r)$ and $v = \gcd(b_1, b_2, m - s)$, by considering equations (7) and (8) and taking greatest common divisors, we have

$$|A|p^r(p^u - 1) = |B|q^s(q^v - 1). \tag{9}$$

Now, by dividing equations (7) and (9), we have

$$\frac{p^{n-r} - 1}{p^u - 1} = \frac{q^{m-s} - 1}{q^v - 1} \tag{10}$$

and by dividing equations (8) and (9), we have

$$\frac{p^{n-a_i-r} - 1}{p^u - 1} = \frac{q^{m-b_i-s} - 1}{q^v - 1}. \tag{11}$$

Note that it is not possible that $n - a_1 - r = u = n - a_2 - r$, since $a_1 \neq a_2$. Now, Theorem 2.3 and equation (10) and (11) yield a contradiction. $\square$

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